Quantum Physics II (8.05) Fall 2004
Assignment 11

Massachusetts Institute of Technology
Physics Department
November 17, 2004

Due WEDNESDAY, November 24
7pm

This week we finish the radial equation. We will then begin our discussion of the addition of angular momenta.

Reading Assignment for week 11 of the course

- (continued from last week) The "factorization method" is discussed in Ch.6 and Ch.8 of Ohanian. Ohanian sets up his notation and proves a general theorem in §6.2. (We will *not* cover the details of this particular theorem in lecture.) Ch.8 contains three important examples of the factorization method, i) the isotropic harmonic oscillator, ii) hydrogen, and iii) the free particle in spherical coordinates.
- Griffiths §4.4, in particular §4.4.3 on the addition of angular momenta.

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Problem Set 11

1. **Coupled One-dimensional Harmonic Oscillators** [5 points]

   Consider two coupled one-dimensional harmonic oscillators with the Hamiltonian

   $$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{m\omega^2}{2}\left[x_1^2 + x_2^2 + 2\lambda(x_1 - x_2)^2\right] \tag{1}$$

   (a) Using a similar procedure to what we did for hydrogen in lecture, define center of mass coordinates P, R and relative coordinates p, r for the two particles. Work out expressions for all the commutation relations for these operators.

   (b) Define a total mass M and reduced mass μ. Write H in term of these mass parameters and the coordinates from part (a). Show that H separates into two parts, where each part is built of operators that commute with all the operators in the other part.

   (c) Determine all the energy eigenvalues for H, and explain what quantum numbers you would use to label the energy eigenstates.

2. **A pair of power law potentials** [15 points]

   Here is an application of the supersymmetric method to a problem in one dimension. Consider the two Hamiltonians generated by the superpotential $\mathcal{W}(x) = gx^3$. [Choose units so that $\hbar = 2m = 1$].

   (a) What two Hamiltonians are related by this choice of $\mathcal{W}(x)$?

   (b) Graph them for various values of g and note how different these potentials look!

   (c) Which of these Hamiltonians has a zero energy ground state? What is its wavefunction?

   (d) Prove that the wavefunction of the zero energy state of the previous part obeys the Schrödinger equation by direct substitution (and differentiation).

   (e) Prove that similar results would be obtained for any power law superpotential with an **odd power of** x.

3. **Isotropic Harmonic Oscillator in Two-Dimensions** [20 points]

   The isotropic harmonic oscillator in two-dimensions for a particle of mass M has the Hamiltonian

   $$H = \frac{1}{2M}\left(p_x^2 + p_y^2\right) + \frac{M\omega^2}{2}\left(x^2 + y^2\right). \tag{2}$$

   If we switch to polar coordinates (ρ, ϕ) and work in the position representation then this Hamiltonian can be written as

   $$H = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right) + \frac{L_z^2}{2M\rho^2} + \frac{M\omega^2\rho^2}{2}, \tag{3}$$

   where $\rho = \sqrt{x^2 + y^2}$ and $L_z = xp_y - yp_x = -i\hbar\,\partial/\partial\phi$.

(a) We readily observe that $[L_z, H] = 0$. Let $|E, m\rangle$ be the simultaneous eigenstates of these operators, so that $H|E, m\rangle = E|E, m\rangle$ and $L_z|E, m\rangle = m\hbar|E, m\rangle$. What form do you expect for ρ and ϕ dependence of the wavefunction $\psi_{E,m}(\rho, \phi) = \langle \rho, \phi|E, m\rangle$? Solve explicitly for the ϕ-coordinate dependence. What are the possible values for m?

(b) Let the radial wavefunction $f(\rho) = u(\rho)/\sqrt{\rho}$ and derive the radial equation for $u(\rho)$. Define a dimensionless coordinate $r = \rho/b$ and a dimensionless energy $\varepsilon_{\nu,m} = aE_{\nu,m}$, so that the radial equation becomes

$$H_m u_{\nu,m}(r) = \varepsilon_{\nu,m} u_{\nu,m}(r)\,, \qquad H_m = -\frac{d^2}{dr^2} + \frac{(m^2 - 1/4)}{r^2} + r^2\,. \tag{4}$$

What are a and b?

(c) Consider the superpotential

$$\mathcal{W}_m(r) = -\frac{(|m| + 1/2)}{r} + r.$$

Determine the Hamiltonians $H_m^{(1)}$ and $H_m^{(2)}$. Relate these Hamiltonians to each other and to H_m.

(d) Let's assume that $H_m^{(1)}$ has an eigenstate with zero energy without constructing it explicitly. [You may construct it if you wish. You should find $u_{0,m}(r) = r^{|m|+1/2}\, e^{-r^2/2}$. Thus the absolute value of m that appears in $\mathcal{W}_m(r)$ ensures that $u_{0,m}(0) = 0$ as desired.] By following the steps of the supersymmetric method as outlined in the handout, show that the energy levels for H_m are

$$\varepsilon_{\nu,m} = 2|m| + 4\nu + 2$$

for integers $\nu \geq 0$. What are the corresponding energy levels $E_{\nu,m}$ for our original Hamiltonian? For the eigenstate with the smallest energy is the answer what you expect? Is the energy and degeneracy of the first excited state also what you expect? [Recall that you already know the answer for the energy levels of the Hamiltonian in (2), see for example problem set 4.]

4. **Analyzing a Set of Potentials Related by Supersymmetry** [20 points]

In this problem we will study another family of Hamiltonians related by operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ defined with the aid of a superpotential, $\mathcal{W}(x)$. Like the hydrogen atom, there is an entire family of Hamiltonians labeled by an index n (it was the orbital angular momentum, ℓ, for hydrogen). Also like the hydrogen case, you have to solve this problem by using the fact that both Hamiltonians, $H_n^{(1)}$ and $H_n^{(2)}$ are related to the Hamiltonian of interest, H_n^{PT}.

If you are not familiar with the hyperbolic functions, eg with relations like $\cosh^2 x - \sinh^2 x = 1$ and $\tanh^2 x = 1 - \mathrm{sech}^2 x$, I would suggest that you work on this problem collaboratively with one of your fellow students who is.

Consider the Schrödinger equation in one dimension, in which we have scaled x and the energy to make everything in the Schrödinger equation dimensionless, and to set $\hbar = 2m = 1$.

The Pöschl-Teller Hamiltonians are defined to be

$$H_n^{\mathrm{PT}} = -\frac{d^2}{dx^2} - n(n+1)\mathrm{sech}^2 x \tag{5}$$

where $n = 0, 1, 2, 3, \ldots$ can be any nonnegative integer.

Your task in this problem is to find energies and eigenstates for all of the bound states for all of these infinitely many Hamiltonians.

(a) Consider the infinite set of superpotentials

$$\mathcal{W}_n = n \tanh x \tag{6}$$

where $n = 0, 1, 2, 3, \ldots$. Using the methods from the supplementary notes, construct $\mathcal{A}_n$, $\mathcal{A}_n^\dagger$, $H_n^{(1)}$ and $H_n^{(2)}$ from the superpotentials (6). Show that

$$H_n^{(1)} = H_{n+1}^{(2)} - (2n+1) \tag{7}$$

Relate $H_n^{(1)}$ and $H_n^{(2)}$ to the Pöschl-Teller Hamiltonians.

(b) Show that $H_n^{(1)}$ has a ground state with energy eigenvalue $E = 0$. Find the ground state wave function for each n. [You need not normalize the wave functions. That is, you only need to find the wave functions up to an overall normalization constant.]

(c) You have shown that $H_1^{(1)}$ has a bound state with energy $E = 0$. Use this fact, the relation (7), and results proved in general in the supplementary notes to conclude that $H_2^{(1)}$ must have a bound state with a particular positive energy. Find this energy eigenvalue. (You have now found two bound states of $H_2^{(1)}$: the ground state that you found in part (b) and the excited state that you have found here.)

(d) Suppose that I tell you that $H_1^{(1)}$ has *only one bound state*, namely the ground state that you found in part (b). Assuming this to be the case, show that $H_n^{(1)}$ has precisely n bound states. Find the energy eigenvalues for all n bound states of $H_n^{(1)}$. Write an expression for the wave function associated with each eigenstate in the form of differential operator(s) acting on wave functions you have found explicitly in part (b).

(e) Consider the Hamiltonian $H_0^{(1)}$. What is the lowest energy eigenvalue of this Hamiltonian? Use this fact to demonstrate that $H_1^{(1)}$ cannot have a second bound state with energy eigenvalue between 0 and 1. [This proves that $H_1^{(1)}$ has only one bound state, its ground state with $E = 0$, as I had you assume above.]

(f) What are all the bound state energies of the Pöschl-Teller Hamiltonians H_n^{PT}?

(g) **This last part is optional and will not be graded, although a solution will be provided.** The Pöschl-Teller potentials also have unbound "scattering" states with positive energy. [Unlike the harmonic oscillator the potential $-n(n+1)\mathrm{sech}^2 x$ goes to zero as $x \to \pm\infty$ and therefore allows for scattering.] You know what the scattering states are for H_0^{PT}: the wave functions are $\exp(ikx)$ with energy k^2. Use the methods developed earlier in this problem to find the wave function for the scattering state with energy k^2 in the potentials $-n(n+1)\mathrm{sech}^2 x$ with $n = 1$ and $n = 2$. Calculate the reflection and transmission coefficients $R_n(k)$ and $T_n(k)$, and show that $R_n = 0$ and $|T_n| = 1$, meaning that these potentials are "reflectionless". [Do the calculation only for $n = 1, 2$. But, the potentials are in fact reflectionless for all n.]